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THE EFFECT OF THE NONLINEAR STRESS–STRAIN RELATIONSHIP ON THE MECHANICAL BEHAVIOR OF OPTICAL GLASS FIBERS

E. SUHIR
AT&T Bell Laboratories, 600 Mountain Avenue, Murray Hill, NJ 07974, U.S.A.

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Abstract—We evaluate the effect of the nonlinear stress–strain relationship on elastic stability, free vibrations, and bending of optical glass fibers. The analysis is carried out under an assumption that this relationship, obtained for the case of uniaxial tension (Mallinder and Proctor, 1964, *Phys. Chem. Glasses* **5**, 91–103; Krause *et al.*, 1979, *Phys. Chem. Glasses* **20**, 135–139; Glaesemann *et al.*, 1988, *11th Opt. Fiber Comm. Conf.* **26**), is also valid in the case of compression, and is applicable to bending deformations as well. We examine low temperature microbending of infinitely long dual coated fibers, elastic stability of short bare fibers, thermally induced stresses and strains in a lightwave coupler, free vibrations of fibers subjected to tension, and bending deformations of fibers experiencing large deflections. We conclude that the nonlinear stress–strain relationship in silica materials can have a significant effect on the mechanical behavior of optical fibers.

INTRODUCTION

It has been found (Mallinder and Proctor, 1964; Krause *et al.*, 1979; Glaesemann *et al.*, 1988) that the stress–strain relationship in glass optical fibers subjected to uniaxial tension is nonlinear and, in the region of strains not exceeding 5%, can be described by the equation

$$\sigma = E_0\varepsilon(1+\tfrac{1}{2}\alpha\varepsilon), \tag{1}$$

where σ is the stress, ε is the strain, E_0 is Young's modulus of the material in the region of very small strains, and α is the parameter of nonlinearity. For most silica materials employed in fiber optics one can assume $E_0 = 72$ GPa, and $\alpha = 6$. The purpose of the analysis which follows is to determine whether the above nonlinear stress–strain relationship can have an appreciable effect on the elastic stability, thermally induced stresses, free vibrations, and bending of optical fibers and lightwave couplers. Accordingly, we evaluate the critical (buckling) stress, thermal stresses and strains, natural frequencies, and maximum stresses in glass fibers, experiencing compression, thermally induced tension, lateral vibrations, and excessive bending, respectively. The analysis is carried out under an assumption that the relationship (1) holds not only for tensile strains, but for compressive strains as well, i.e. can be simply extended into the region of compressive strains as follows:

$$\sigma = E_0\varepsilon(1-\tfrac{1}{2}\alpha\varepsilon). \tag{2}$$

ANALYSIS

Elastic stability

Elastic instability (buckling) of optical fibers can adversely affect both their long-term reliability and added transmission losses (Gloge, 1972; Gardner, 1975; Katsuyama *et al.*, 1980; Suhir, 1988a–c). In this section we assess the effect of the stress–strain relationship (2) on the critical (buckling) stress in infinitely long dual-coated fibers, and in very short bare fibers.

Microbending of dual-coated fibers. The critical force, causing low temperature microbending of dual-coated fibers with compliant (thick and low modulus) primary coatings, can be determined by the formula (Suhir, 1988a):

$$T_c = r_0^2\sqrt{\pi KE}. \tag{3}$$

Here r_0 is the glass fiber radius, K is the spring constant of the coating system (Suhir, 1988c), and E is Young's modulus of the silica material. The formula (3) can be written as

$$\sigma_c = \sqrt{\frac{1}{\pi}KE}, \tag{4}$$

where

$$\sigma_c = \frac{T_c}{\pi r_0^2} \tag{5}$$

is the critical stress. In the formula (4), Young's modulus E is strain (stress) dependent and is therefore a function of the stress σ. This dependency can be found from (2) by differentiation:

$$E = \frac{d\sigma}{d\varepsilon} = E_0(1-\alpha\varepsilon). \tag{6}$$

Then the formula (4) can be written as:

$$\sigma_c = \eta_1\sigma_0, \tag{7}$$

where

$$\sigma_0 = \sqrt{\frac{1}{\pi}KE_0} \tag{8}$$

is the "nominal" critical stress, determined on the basis of "linear" (low strain) Young's modulus E_0, and the factor

$$\eta_1 = \sqrt{1-\alpha\varepsilon} \tag{9}$$

considers the effect of the nonlinear stress–strain relationship. Putting in (2) $\sigma = \sigma_c$, solving the obtained equation for the strain ε, and substituting this solution into (9), we obtain the following equation for the factor η_1:

$$\eta_1^4 + 2\bar{\alpha}\eta_1 - 1 = 0, \tag{10}$$

where

$$\bar{\alpha} = \alpha\frac{\sigma_0}{E_0}. \tag{11}$$

The numerical solution to eqn (10) is plotted for $\alpha = 6$ in Fig. 1.

The calculations carried out in Suhir (1988b) for a fiber with a dual acrylate coating system resulted in a spring constant value equal to $K = 3046$ MPa. With $E_0 = 72$ GPa, we obtain: $\sigma_0 = 8339$ MPa $= \sigma_0/E_0 = 0.1158$, and $\eta_1 = 0.595$. Hence, consideration of the nonlinear stress–strain relationship resulted in a significant reduction in the critical stress. It should be pointed out, however, that the strain corresponding to the calculated stress σ_0 is as high as 7.9%, that is above the 5% value, for which the experimental relationship (1) was obtained. Therefore the actual magnitude of the factor η_1 can be somewhat higher than the calculated value.

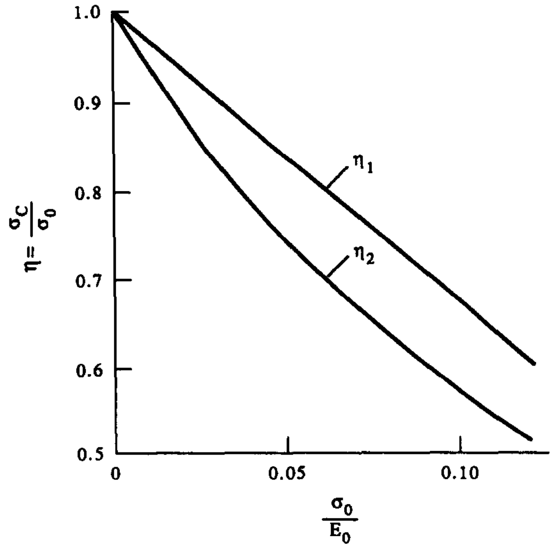

Fig. 1. Effect of the nonlinear stress–strain relationship on the critical stress in coated (η_1) and bare (η_2) fibers.

For a silicone/nylon coating system, the calculated spring constant is only $K = 90$ MPa. Then $\sigma_0 = 1433$ MPa, $\sigma_0/E_0 = 0.0198$, and $\eta_1 = 0.932$. In this case, because of a very compliant coating, the effect of the nonlinear stress–strain relationship is substantially smaller. Since the corresponding strain is only 1.8%, the obtained result is thought to be accurate enough.

Buckling of a short bare fiber. Examine now a short bare optical fiber clamped at its ends and subjected to compressive axial loading. Such a situation can occur, for instance, in a fiber located in a termination fixture (Suhir, 1988c). We proceed from the following Euler formula for the critical force [see, for instance, Timoshenko and Gere (1961) and Suhir (1991)]:

$$T_c = \frac{\pi^2 EI}{(\mu l)^2}. \tag{12}$$

Here l is the length of the fiber, $I = (\pi/4)r_0^4$ is the moment of inertia of the fiber cross-sectional area, μ is a factor depending on the boundary conditions at the supports, and E is Young's modulus of the material. The formula (12) can be written as

$$\sigma_c = \left(\frac{\pi r_0}{2\mu l}\right)^2 E, \tag{13}$$

where the critical stress σ_c is related to the force T_c by (5). Using the relationship (6) for Young's modulus, we obtain the formula (13) in the form (7), where the "nominal" (linear) stress σ_0 is expressed as

$$\sigma_0 = \left(\frac{\pi r_0}{2\mu l}\right)^2 E_0, \tag{14}$$

and the factor

$$\eta_2 = 1 - \alpha\varepsilon \tag{15}$$

considers the effect of the nonlinear stress–strain relationship on the critical stress σ_c. Solving eqn (2) for the strain ε, and substituting the obtained expression in (15), we obtain:

$$\eta_2 = \sqrt{1+\bar{\alpha}^2} - \bar{\alpha}, \tag{16}$$

where the parameter $\bar{\alpha}$ is expressed by (11). The factor η_2 is plotted in Fig. 1 as a function of the ratio of the "nominal" critical stress (calculated without considering nonlinear stress–strain relationship) to the "nominal" (low strain) Young's modulus.

If, for instance, a 2 mm long fiber is clamped at its ends ($\mu = 0.5$), then $\sigma_0 = 696$ MPa, $\sigma_0/E_0 = 0.0096$, $\bar{\alpha} = 0.0578$, and $\eta_2 = 0.9439$. For a 1 mm long fiber, the calculated factor η_2 is substantially smaller: $\eta_2 = 0.7951$. Thus, for very short fibers, the nonlinear stress–strain relationship can have a significant effect on the critical stress and should be accounted for.

Stresses and strains in a lightwave coupler subjected to tension

In fused biconical taper (FBT) couplers, the cores of the fibers are positioned very close to each other, so that the two fundamental modes become coupled through their evanescent fields (Sheem and Giallorenzi, 1979; Sheem and Cole, 1979; Bergh *et al.*, 1980; Villaruel and Moeller, 1981; Bures *et al.*, 1983; Bilodeau *et al.*, 1987). In order to bring the cores of the fibers in close proximity, the cladding in the midportion of the coupler has to be made very thin (Fig. 2). At the same time, the coupler must be sufficiently strong, both on a short and a long-time scale, and must be able to withstand appreciable axial deformations. These are usually caused by its thermal contraction mismatch with the substrate, but could be applied also deliberately to improve the dynamic stability of the coupler (by increasing is natural frequencies of vibration).

In this section we evaluate the stresses and strains in a FBT coupler subjected to tensile deformations, with consideration of its nonprismaticity. We assume that the actual coupler geometry can be approximated by two circular conical parts connected by a circular cylindrical midportion (as shown in Fig. 2 in broken lines). Such an approximation is thought to be adequate, as long as the radii r_c and r_f are chosen in such a way that the areas of the corresponding circles are equal to the actual cross-sectional areas.

The stress $\sigma(x)$ in any cross-section x of the coupler can be evaluated as

$$\sigma(x) = \frac{P}{\pi r^2(x)} = \sigma_f \frac{r_f^2}{r^2(x)}, \tag{17}$$

where r_f^2 is the radius of the fused midportion, $\sigma_f = P/\pi r_f^2$ is the stress experienced by this portion, P is the applied tensile force, and the radius $r(x)$ of the coupler's cross-section is changing, in accordance with the assumption, as follows:

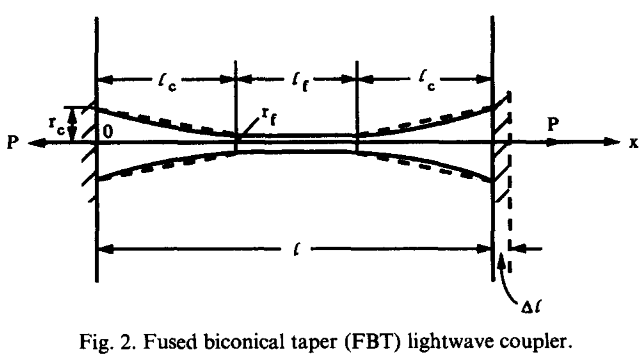

Fig. 2. Fused biconical taper (FBT) lightwave coupler.

$$r = \begin{cases} r_c-(r_c-r_f)\dfrac{x}{l_c}, & 0 \leqslant x \leqslant l_c, \\ r_f, & l_c \leqslant x \leqslant l_c+l_f, \\ r_c-(r_c-r_f)\dfrac{l-x}{l_c}, & l_c+l_f \geqslant x \leqslant l. \end{cases} \tag{18}$$

Here l, l_c and l_f are the total length of the coupler, the length of one of its conical parts, and the length of its fused midportion, respectively, r_c is the larger radius of the conical part, and r_f is the radius of the fused midportion. The origin of the coordinate x is at the left end of the coupler. From (1) and (17) we obtain:

$$\varepsilon = \frac{1}{\alpha}\left(\sqrt{1+2\alpha\frac{\sigma}{E_0}}-1\right) = \frac{1}{\alpha}\left(\sqrt{1+\beta^2\frac{r_f^2}{r^2}}-1\right), \tag{19}$$

where the parameter

$$\beta = \sqrt{2\alpha\frac{\sigma_f}{E_0}} = \sqrt{2\alpha\frac{P}{\pi E_0 r_f^2}} \tag{20}$$

considers the effect of the magnitude of the applied force and the nonlinear behavior of the material on the strain.

The total elongation Δl of the coupler is

$$\Delta l = \int_0^l \varepsilon(x)\,\mathrm{d}x = \frac{1}{\alpha}\left(\int_0^l \sqrt{1+\beta^2\frac{r_f^2}{r^2}}\,\mathrm{d}x - l\right). \tag{21}$$

The integral in the obtained expression can be written as

$$\int_0^l \sqrt{1+\beta^2\frac{r_f^2}{r^2}}\,\mathrm{d}x = 2\int_0^{l_c}\sqrt{1+\beta^2\frac{r_f^2}{r^2}}\,\mathrm{d}x + \sqrt{1+\beta^2}\,l_f = 2f_c l_c + f_f l_f, \tag{22}$$

where the factors

$$f_c = \frac{1}{l_c}\int_0^{l_c}\sqrt{1+\beta^2\frac{r_f^2}{r^2}}\,\mathrm{d}x \tag{23}$$

and

$$f_f = \sqrt{1+\beta^2} \tag{24}$$

consider the effects of the magnitude of the applied force and the material's nonlinearity on the elongations of the conical and the fused portions, respectively. The factor f_c can be evaluated, taking into account the first relationship in (18), as follows:

$$\begin{aligned} f_c &= \frac{1}{l_c}\int_0^{l_c}\sqrt{1+\beta^2\frac{r_f^2}{r^2}}\,\mathrm{d}x = \frac{1}{r_c-r_f}\int_{r_f}^{r_c}\sqrt{1+\beta^2\frac{r_f^2}{r^2}}\,\mathrm{d}r \\ &= \frac{1}{1-\rho}\left(\sqrt{1+\beta^2\rho^2}-\rho\sqrt{1+\beta^2}+\beta\rho\ln\frac{\sqrt{1+\beta^2}+\beta}{\sqrt{1+\beta^2\rho^2}+\beta\rho}\right), \end{aligned} \tag{25}$$

where $\rho = r_f/r_c$ is the radii ratio.

Table 1. Calculated tensile free in a lightwave coupler

β	0	0.02	0.04	0.06	0.1	0.2	0.4	0.5	0.6	0.7	0.8
f_f	1	1.0002	1.0008	1.0018	1.0050	1.0198	1.0770	1.1180	1.1662	1.2207	1.2806
f_c	1	1.00000	1.00006	1.00013	1.00036	1.00296	1.00631	1.00979	1.01397	1.01886	1.02429
$\varepsilon_0 = \sigma_f/E_0$, %	0	0.0033	0.0133	0.0300	0.0833	0.3333	1.3333	2.0833	3.0000	4.0833	5.3333
$e_f = (f_f - 1)/\alpha$, %	0	0.0033	0.0133	0.0300	0.0833	0.3300	1.2833	1.9667	2.7700	3.6783	4.6767
$\Delta l/l$, %	0	0.0010	0.00466	0.00985	0.0291	0.1332	0.4571	0.7019	0.9908	1.3193	1.6810
P, gf	0	0.0773	0.3093	0.6959	1.933	0.732	30.93	48.33	69.59	94.72	123.72
		(0.0654)	(0.3047)	(0.6440)	(1.903)	(8.709)	(29.89)	(45.89)	(64.78)	(86.26)	(109.91)

The calculated values of the factors f_f and f_c (for $\rho = 0.08$) are shown in Table 1. As is evident from this table, the factors f_f, reflecting the effect of the fused midportion on the total elongation of the coupler, are substantially larger than the factors f_c, considering the effect of the conical parts. This is due to the relatively high compliance of the fused midportion. Table 1 shows also that, for sufficiently large deformations, the "actual" (nonlinear) strains ε_f in the fused midportion, calculated for the given stress σ_f by the formula (17), are appreciably smaller than the "nominal" strains $\varepsilon_0 = \sigma_f/E_0 = \beta^2/2\alpha$, predicted by the linear theory.

From (21) and (22) we obtain:

$$\frac{\Delta l}{l} = \frac{1}{\alpha}\left[\frac{2l_c}{l} f_c(\beta, \rho) + \frac{l_f}{l} f_f(\beta) - 1\right]. \tag{26}$$

This equation can be used to evaluate the parameter β, and then the axial force P, for the given total elongation Δl. The calculated forces P and the total strains $\Delta l/l$, shown in Table 1, were obtained for the case $l = 38.5$ mm, $l_f = 11.5$ mm, $r_f = 0.01$ mm, and $r_c = 0.125$ mm. As is evident from the calculated data, rather low total strains $\Delta l/l$ lead to significantly higher strains ε_f in the fused midportion. This is also due to the fact that the conical parts are substantially more rigid than the fused midportion and therefore add much less to the total elongation of the coupler. Let, for instance, the total displacement of the above coupler be $\Delta l = 0.0513$ mm, so that the total strain is $\Delta l/l = 0.1332\%$. As follows from Table 1, in this case, the strain in the fused midportion is $\varepsilon_f = 0.33\%$, and its elongation is $\Delta l_f = \varepsilon_f l_f = 0.0380$ mm. Hence, the conical parts, whose total length is 27.0 mm, i.e. by a factor of 2.35 greater than the length of the fused midportion, are stretched by $\Delta l_c = 0.0513 - 0.0380 = 0.0133$ mm, which is only a little more than 25% of the total elongation. The stresses in the fused portion of the coupler can be easily determined from the calculated σ_f/E_0 values. If, for instance, the total strain in the coupler is 0.02%, the strain in the fused midportion of the coupler is about 0.054%, and the corresponding stress is $\sigma_f = 0.00054 \times 7384 = 4.0$ kg mm^{-2}.

If the nonlinear stress–strain relationship was not considered, then eqn (26) would result in the following formula for the applied force:

$$P = \frac{\pi E_0 r_f^2 \Delta l}{2\rho l_c + l_f}.$$

The P values, calculated on the basis of this formula for the $\Delta l/l$ values obtained in Table 1 are shown in parentheses in the bottom line of this table. As evident from the calculated data, the linear approach can result in a substantial underestimation of the force P, and, hence, of the tensile stresses.

Free vibrations

Long fiber subjected to tension. From the standpoint of structural analysis, a long-and-thin fiber subjected to tension can be treated as a spring (or a thread). Then its vibrations can be described by the equation:

$$P\frac{\partial^2 w}{\partial x^2}-m\frac{\partial^2 w}{\partial t^2}=0. \tag{27}$$

Here $w = w(x, t)$ is the deflection function, P is the tensile force, m is the fiber's mass per unit length, x is the longitudinal coordinate, and t is time. Equation (27) assumes that the fiber is so long and its diameter is so small, that the flexural rigidity of the fiber need not be considered. This equation simply states that the inertia forces must be equilibrated by the elastic tensile force in the fiber.

We present the function w in the form of a series:

$$w=\sum_{i=1}^{\infty}\theta_i(t)\sin\frac{i\pi x}{l}, \tag{28}$$

where l is the length of the fiber, and $\theta_i(t)$ is the principal coordinate of the ith mode of vibrations. Substituting (28) into (27), we obtain:

$$\ddot{\theta}_i+\omega_i^2\theta_i=0,\quad i=1,2,\dots$$

Here

$$\omega_i=\frac{i\pi}{l}\sqrt{\frac{P}{m}}=\frac{i\pi}{l}\sqrt{\frac{\sigma}{\rho}},\quad i=1,2,\dots \tag{29}$$

is the vibration frequency of the ith mode, σ is the tensile stress in the fiber, and ρ is the density of the material (typically, $\rho = 2.2\ \mathrm{g\ cm^{-3}}$). Substituting (1) into (29), we have:

$$\omega_i=\eta_3\omega_i^0, \tag{30}$$

where

$$\omega_i^0=\frac{i\pi}{l}\sqrt{\frac{E_0\varepsilon}{\rho}} \tag{31}$$

is the "nominal" (linear) frequency, and the factor

$$\eta_3=\sqrt{1+\tfrac{1}{2}\alpha\varepsilon} \tag{32}$$

considers the effect of the nonlinear stress–strain relationship. If, for instance, the actual strain in the fiber is $\varepsilon = 0.05$, then the formula (32) yields: $\eta_3 = 1.072$. Thus, consideration of the nonlinear stress–strain relationship increased the vibration frequencies by about 7%.

Vibration frequency of a lightwave coupler. The total energy of the free vibrations of the coupler structure (Fig. 2) is due to its kinetic energy

$$T=\frac{1}{2}\int_0^l m(x)\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x \tag{33}$$

and the strain energy

$$V=\tfrac{1}{2}P\int_0^l\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x+\frac{1}{2}\int_0^l EI(x)\left(\frac{\partial^2 w}{\partial x^2}\right)^2\mathrm{d}x. \tag{34}$$

Here $w = w(x, t)$ are the lateral deflections of the coupler, $m(x) = \pi(\gamma/g)r^2(x)$ is the mass of the coupler per unit length, $\gamma/g = 2.245\times10^{-10}\ \mathrm{kg\ s^2\ mm^{-4}}$ is the density of the coupler's material, γ is its specific weight, g is the acceleration due to gravity, and $I(x) = (\pi/4)r^4(x)$ is the moment of inertia of the cross-sectional area. The formula (34) reflects an assumption that the vibration amplitudes are small and therefore the additional strain energy due to the axial deformations caused by lateral deflections need not be considered.

Actual lightwave couplers are characterized by very large lengths compared to their lateral dimensions (even at the end cross-sections), and therefore the actual boundary conditions at the ends have a small effect on the vibrations. Indeed, consider, for the sake of simplicity, a uniform beam clamped at the ends and subjected to a tensile force P. The vibrations of this beam can be described by the equation

$$EI\frac{\partial^4 w}{\partial x^4} - P\frac{\partial^2 w}{\partial x^2} + m\frac{\partial^2 w}{\partial t^2} = 0. \tag{35}$$

Seeking the deflection function $w(x, t)$ in the form of an expansion

$$w(x, t) = \sum_{i=1}^{\infty} X_i(x) \sin \omega_i t, \tag{36}$$

where ω_i is the frequency of the ith mode, we find that the functions $X_i(x)$ can be determined from the equation

$$EIX_i^{IV}(x) - PX_i''(x) - m\omega_i^2 X_i(x) = 0. \tag{37}$$

This equation has the following solution:

$$X_i(x) = A_i \cosh \gamma_i x + B_i \sinh \gamma_i x + C_i \cos \delta_i x + D_i \sin \delta_i x, \tag{38}$$

where

$$\gamma_i = \sqrt{\frac{P}{2EI}(\sqrt{1+\varepsilon_i} - 1)}, \quad \delta_i = \sqrt{\frac{P}{2EI}(\sqrt{1+\varepsilon_i} + 1)}, \tag{39}$$

and

$$\varepsilon_i = \frac{4EI}{P^2} m\omega_i^2. \tag{40}$$

Note that

$$\gamma_i \delta_i = \omega_i \sqrt{\frac{m}{EI}}, \quad \delta_i^2 - \gamma_i^2 = \sqrt{\frac{P}{EI}}. \tag{41}$$

After substituting (38) into the boundary conditions

$$X_i(0) = X_i(l) = 0, \quad X_i'(0) = X_i'(l) = 0$$

and considering (41), we obtain the following equation for the vibration frequencies ω_i:

$$2\gamma_i \delta_i (\cosh u_i \cos v_i - 1) + (\delta_i^2 - \gamma_i^2) \sinh u_i \sin v_i = 0. \tag{42}$$

Here $u_i = \gamma_i l$, $v_i = \delta_i l$, and the parameters γ_i and δ_i are expressed through the frequencies ω_i by the formulae (39) and (40). If the ε_i value is small compared to unity, i.e. if the actual tensile force P is significantly larger than the value

$$P_s = 2\omega_i\sqrt{EIm}, \tag{43}$$

then the formulae (39) yield: $\gamma_i = 0$, $\delta_i = \sqrt{P/EI}$, and eqn (42) reduces to the frequency equation for a simply-supported bar:

$$\sin v_i = 0. \tag{44}$$

In this case the vibration modes can be presented as

$$X_i(x) = \sin\frac{i\pi x}{l} \tag{45}$$

and eqn (37) results in the following formula for the frequency:

$$\omega_i = \frac{i\pi}{l}\sqrt{\frac{P}{m}+\left(\frac{i\pi}{l}\right)^2\frac{EI}{m}}. \tag{46}$$

With this formula, the expression (43) can be written as

$$P_s = 2(1+\sqrt{2})\left(\frac{i\pi}{l}\right)^2 EI = i^2\frac{\pi^3(1+\sqrt{2})}{2}E\frac{r_0^4}{l^2}. \tag{47}$$

Assuming in the example, considered above, $r_0 = (r_c+r_f)/2 = 0.0675$ mm, $E = E_0$, and $i = 1$, we obtain: $P_s = 0.00387$ fg. Hence, as one can see from Table 1, the coupler structure in this example can be considered simply supported at the ends even for rather low values of the force P.

Let us assess now the second (bending) term in (34) compared to the first term. Let us assume again, for the sake of simplicity, that the flexural rigidity EI is constant and is equal to $(\pi/4)E_0r_0^4$. Then, presenting the deflection function w in the form (36), where the vibration mode function $X_i(x)$ is expressed by (45), we obtain:

$$V = \frac{Pl}{4}\sum_{i=1}^{\infty}\left(\frac{i\pi}{l}\right)^2\left(1+i^2\frac{\pi^3}{2}\frac{Er_0^4}{Pl^2}\right)A_i^2\sin^2\omega_i t,$$

where A_i is the amplitude of the ith mode. As follows from this formula, the bending term contributes very little to the strain energy, if the tensile force P is significantly larger than the value

$$P_b = i^2\frac{\pi^3}{2}E\frac{r_0^4}{l^2}.$$

Comparing this formula with (47), we conclude that if the condition $P \gg P_s$ is fulfilled, the condition $P \gg P_b$ is fulfilled as well, i.e. if the tensile force P is large enough, so that the coupler can be considered simply supported at it ends, it is also sufficiently large for neglecting the bending term in the expression for the strain energy. Hence, for a large enough tensile force ($P \gg P_s$), the strain energy can be assessed by the simplified formula:

$$V = \tfrac{1}{2}P\int_0^l\left(\frac{\partial w}{\partial x}\right)^2 dx. \tag{48}$$

For the lowest vibration mode ($i = 1$), the formulae (33) and (48) yield

$$T = \frac{1}{2}\frac{\gamma}{g} A^2\omega^2 \cos^2 \omega t \int_0^l r^2(x) \sin^2 \frac{\pi x}{l}\, dx$$
$$= \frac{\gamma}{g} A^2\omega^2 l^3 C \cos^2 \omega t,$$
$$V = \frac{\pi^2}{4}\frac{P}{l} A^2 \sin^2 \omega t,$$

where A is the amplitude of vibrations and the constant C is expressed as follows:

$$C = \left(\frac{r_c - r_f}{l_c}\right)^2 \left\{\frac{1}{6}\left(\frac{l_c}{l}\right)^3 + \left[\frac{1}{8\pi^3} - \frac{1}{4\pi}\left(\frac{l_c}{l}\right)^2\right] \sin 2\pi \frac{l_c}{l} - \frac{1}{4\pi^2}\frac{l_c}{l} \cos 2\pi \frac{l_c}{l}\right\}$$
$$-2\frac{r_c}{l}\frac{r_c - r_f}{l_c}\left[\frac{1}{8\pi^2} + \frac{1}{4}\left(\frac{l_c}{l}\right)^2 - \frac{1}{4\pi}\frac{l_c}{l} \sin 2\pi \frac{l_c}{l} - \frac{1}{8\pi^2} \cos 2\pi \frac{l_c}{l}\right]$$
$$+\frac{1}{2}\left(\frac{r_c}{l}\right)^2\left(\frac{l_c}{l} - \frac{1}{2\pi} \sin 2\pi \frac{l_c}{l}\right) + \frac{1}{4}\left(\frac{r_f}{l}\right)^2\left[\frac{l_f}{l} + \frac{1}{2\pi} \sin 2\pi \frac{l_c}{l} - \frac{1}{2\pi} \sin 2\pi \frac{l_c + l_f}{l}\right].$$

The condition $T_{max} = V_{max}$ results in the following simple formula for the vibration frequency:

$$\omega = \frac{\pi}{2l^2}\sqrt{\frac{g}{\gamma}\frac{P}{C}}. \tag{49}$$

From this formula we conclude that the initial strain resulting in the desired (or required) lowest vibration frequency ω, causes the following stress in the fused midportion of the coupler:

$$\sigma_f = 2\frac{\gamma}{g}\frac{C}{\pi^3}\left(\frac{\omega l^2}{r_f}\right)^2. \tag{50}$$

Let the highest expected excitation frequency be, say, 2000 Hz. With the factor of safety equal to two, the required natural frequency of the coupler vibration is 4000 Hz. Then, for the coupler, examined earlier, we obtain: $C = 16.866 \times 10^{-8}$, $\sigma_f = 67.8$ kg mm^{-2}, $\varepsilon_f = 0.894\%$. Thus, the silica material should be sufficiently strong to withstand long-term strains higher than 0.9%. Note that the formula (50) can be used also for the prediction of the initial tensile force, stress and strain from the measured frequency of vibrations.

The tensile force P, causing this stress, is $P = 21.3$ g. If the nonlinear stress–strain relationship were not considered, then, according to the Table 1 data, the tensile force would be about $P = 20.7$ g. Since, as follows from (49), the vibration frequency is proportional to the square root of the tensile force, the linear frequency would be lower than the nonlinear frequency only by a factor of 0.986. Hence, in this case the effect of the nonlinear stress–strain relationship need not be taken into account.

Bending

Basic equations. The position of the centroid (neutral axis) of an optical fiber subjected to bending can be determined from the equation (Fig. 3):

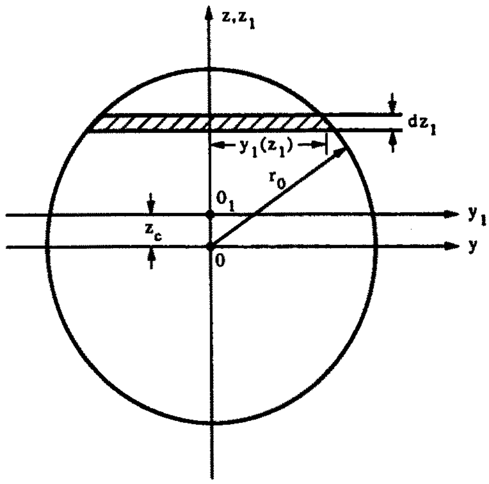

Fig. 3. Cross-section of a fiber subjected to bending: z_c, deviation of the neutral axis from the geometrical center of the fiber cross-section.

$$\int_A E z_1 \, \mathrm{d}A = 0, \tag{51}$$

where E is Young's modulus of the material, and A is the cross-sectional area. An element of this area can be evaluated by the formula:

$$\mathrm{d}A = \sqrt{r_0^2 - (z_1 + z_c)^2} \, \mathrm{d}z_1. \tag{52}$$

Since the strain ε is related to the radius R of curvature of the cross-section as

$$\varepsilon = \frac{z_1}{R},$$

the formula (6) for Young's modulus can be written in the case of tension in the form:

$$E = E_0\left(1 + \alpha \frac{z_1}{R}\right). \tag{53}$$

In the analysis which follows we assume that the effect of the nonlinear stress–strain relationship on the radius R of the fiber bend is small compared to its effect on the maximum stress and need not be taken into account. Then, substituting (52) and (53) into (51), we obtain

$$\int_{-(r_0+z_c)}^{r_0-z_c} \left(z_1 + \alpha \frac{z_1^2}{R}\right) \sqrt{r_0^2 - (z_1 + z_c)^2} \, \mathrm{d}z_1 = 0. \tag{54}$$

Introduce a new variable ξ, so that

$$z_1 + z_c = r_0 \sin \xi. \tag{55}$$

Then eqn (54) yields:

$$-z_c\left(1-\frac{\alpha}{R}z_c\right)\int_{-\pi/2}^{\pi/2}\cos^2\xi\,d\xi+\frac{\alpha}{R}r_0^2\int_{-\pi/2}^{\pi/2}\sin^2\xi\cos^2\xi\,d\xi$$
$$+r_0\left(1-2\frac{\alpha}{R}z_c\right)\int_{-\pi/2}^{\pi/2}\sin\xi\cos^2\xi\,d\xi=0.$$

The first two integrals in this equation are equal to $\pi/2$, and $\pi/8$, respectively, and the third integral is zero. Then we obtain the following quadratic equation for the deviation z_c of the neutral axis from the geometrical center of the fiber cross-section:

$$z_c^2-\frac{R}{\alpha}z_c+\frac{r_0^2}{4}=0.$$

This equation has the following solution:

$$z_c=r_0\frac{1-\sqrt{1-(\alpha\varepsilon_0)^2}}{2\alpha\varepsilon_0},\tag{56}$$

where

$$\varepsilon_0=\frac{r_0}{R}\tag{57}$$

is the "nominal" (linear) strain. When this strain is very small, so that the $\alpha\varepsilon_0$ value is substantially smaller than unity, the formula (56) can be simplified as follows:

$$z_c=\tfrac{1}{4}\alpha\varepsilon_0 r_0,\tag{58}$$

and, with $\alpha=6$, yields:

$$\frac{z_c}{r_0}=\tfrac{3}{2}\varepsilon_0.\tag{59}$$

Hence, the relative shift in the centroid due to the nonlinear stress–strain relationship is greater by a factor of 1.5 than the nominal bending strain.

Let us show that in an approximate analysis the flexural rigidity of the fiber cross-section can be indeed assumed constant, i.e. calculated without considering the nonlinear stress–strain relationship. The flexural rigidity of such a cross-section can be expressed by the integral

$$EI=\int_A Ez_1^2\,dA=E_0\int_{-(r_0+z_c)}^{r_0-z_c}\left(z_1^2+\alpha\frac{z_1^3}{R}\right)\sqrt{r_0^2-(z_1+z_c)^2}\,dz_1$$
$$=\left(2E_0r_0^4-3\alpha\frac{E_0r_0^4}{R}z_c\right)\int_{-\pi/2}^{\pi/2}\sin^2\xi\cos^2\xi\,d\xi$$
$$+\left(2E_0r_0^2z_c^2-\alpha\frac{E_0r_0^2}{R}z_c^3\right)\int_{-\pi/2}^{\pi/2}\cos^2\xi\,d\xi+\alpha\frac{E_0r_0^5}{R}\int_{-\pi/2}^{\pi/2}\sin^3\xi\cos^2\xi\,d\xi$$
$$-\left(4E_0r_0^3z_c-3\alpha\frac{E_0r_0^3}{R}z_c^2\right)\int_{-\pi/2}^{\pi/2}\sin\xi\cos^2\xi\,d\xi.$$

The first two integrals in this expression are equal to $\pi/8$ and $\pi/2$, respectively, and the last two integrals are zero. Then we have:

$$EI = \eta_E E_0 I_0, \tag{60}$$

where

$$E_0 I_0 = \frac{\pi}{4} E_0 r_0^4 \tag{61}$$

is the "nominal" (low strain) flexural rigidity, and the factor

$$\eta_E = 1 - \tfrac{3}{2}\alpha\varepsilon_0 \frac{z_c}{r_0} + 4\left(\frac{z_c}{r_0}\right)^2 - 2\alpha\varepsilon_0\left(\frac{z_c}{r_0}\right)^3$$

considers the effect of the stress–strain nonlinearity. Substituting the z_c value from (56) into this equation, we obtain:

$$\eta_E = \frac{2-(2-\alpha^2\varepsilon_0^2)\sqrt{1-\alpha^2\varepsilon_0^2}}{2\alpha^2\varepsilon_0^2}. \tag{62}$$

When the nominal strain is very small, so that the $\alpha^2\varepsilon_0^2$ value is considerably smaller than unity, the formula (62), with $\alpha = 6$, can be simplified as follows:

$$\eta_E \cong 1 - \tfrac{1}{8}(\alpha\varepsilon_0)^2 = 1 - \tfrac{9}{2}\varepsilon_0^2. \tag{63}$$

Comparing (63) with (59), we conclude that the deviation of the "actual" (finite strain) flexural rigidity from its "nominal" (low strain) value is proportional to the "nominal" (linear) strain squared, while the deviation of the centroid (the neutral axis) from the geometrical center-line of the fiber cross-section is proportional to the first power of the "nominal" strain. Therefore, in an approximate analysis, the flexural rigidity of the fiber cross-section can be simply assumed strain independent. This means that the geometry of the fiber bend, and, particularly, the radii of curvature, can be assessed without taking into consideration the nonlinear behavior of the material.

We would like to point out that the formulae (56) and (62) indicate that the basic equations (1) and (2), obtained experimentally for strains not exceeding 5%, cannot be used in the case of very high strains. Indeed, these formulae do not make sense for $(\alpha\varepsilon_0)^2$ values, larger than unity. This corresponds to the nominal strains ε_0 exceeding $\frac{1}{6} = 16.7\%$. Although such high strains cannot occur in regular silica fibers, high strength fibers can withstand even higher strains.

The maximum tensile and the maximum compressive strains in a glass fiber subjected to bending can be determined, with consideration of the shift in the centroid due to the nonlinear stress–strain relationship, as follows:

$$\left.\begin{aligned} \varepsilon_t &= \frac{r_0 - z_c}{R} = \varepsilon_0\left(1 - \frac{1-\sqrt{1-\alpha^2\varepsilon_0^2}}{2\alpha\varepsilon_0}\right) \\ \varepsilon_c &= \frac{r_0 + z_c}{R} = \varepsilon_0\left(1 + \frac{1-\sqrt{1-\alpha^2\varepsilon_0^2}}{2\alpha\varepsilon_0}\right) \end{aligned}\right\}. \tag{64}$$

Then the corresponding maximum stresses can be evaluated, using the relationships (1) and (2), by the formulae:

Table 2. Stresses and strains in optical fibers subjected to two-point bending with consideration of the nonlinear stress–strain relationship

ε_0, %	1	2	3	4	5	6	7	8
ε_t, %	0.985	1.940	2.864	3.7564	4.6162	5.4413	6.2294	6.997
σ_t/E_0	0.01014	0.02053	0.03110	0.04180	0.05255	0.06330	0.07393	0.08437
ε_c, %	1.015	2.060	3.136	4.2436	5.3838	6.5587	7.7707	9.023
σ_c/E_0	0.00984	0.01933	0.02841	0.03703	0.04514	0.05268	0.05959	0.06580
σ_t^0/E_0	0.01030	0.02120	0.03270	0.04480	0.05750	0.07080	0.08470	0.09920
σ_c^0/E_0	0.00970	0.01880	0.02730	0.03520	0.04250	0.04920	0.0553	0.0608
D, mm	14.98	7.49	4.99	3.74	3.00	2.50	2.14	1.87

$$\sigma_t = E_0\varepsilon_t(1+\tfrac{1}{2}\alpha\varepsilon_t), \quad \sigma_c = E_0\varepsilon_c(1-\tfrac{1}{2}\alpha\varepsilon_c).$$

Stresses calculated in accordance with these formulae are presented in Table 2. In this table, ε_0 is the "nominal" (linear) strain, ε_t is the maximum tensile strain, ε_c is the maximum compressive strain, σ_t/E_0 and σ_c/E_0 are the ratios of the maximum tensile and the maximum compressive stresses to the "nominal" (low strain) Young's modulus, and σ_t^0/E_0 and σ_c^0/E_0 are similar ratios calculated by the above formulae, assuming $\varepsilon_t = \varepsilon_c = \varepsilon_0$, that is without considering the effect of the nonlinear stress–strain relationship.

Two-point bending. The obtained results can be particularly helpful for the evaluation of the maximum stresses in optical fibers subjected to large deformations during two-point bending. This technique (Murgatroyd, 1964; France *et al.*, 1980; Cowap and Brown, 1984; Mathewson *et al.*, 1986) involves constraining a bent loop of fiber between two faceplates which are then brought together until the desirable gap is achieved, or until the fiber breaks. A modification of this technique involves inserting a U-shaped bend of a fiber into a glass tube of the given inner diameter. The minimum radius R of curvature at the midpoint of the fiber bend is related to the distance D between the fiber axes in the region beyond the bend as follows:

$$R = \frac{1}{2}\frac{1/\sqrt{2}}{2E(1/\sqrt{2})-K(1/\sqrt{2})}D = 0.4173D. \tag{65}$$

Here $K(p)$ and $E(p)$ are complete elliptic integrals of the first and the second kind, respectively. The relationship (65) can be easily obtained as a special case of the well-known solution to the "elastica problem" [see, for instance, Timoshenko and Gere (1961) and Suhir (1991)], and results in the following formula for the "nominal" strain:

$$\varepsilon_0 = \frac{r_0}{R} = 2.3964\frac{r_0}{D} \cong 2.4\frac{r_0}{D}. \tag{66}$$

The calculated D values are shown for a 125 μm glass fiber ($r_0 = 0.0625$ mm) in Table 2, and the ratios of the induced stresses to "nominal" (low strain) Young's modulus are plotted in Fig. 4. As evident from these data, the nonlinear stress–strain relationship can have an appreciable effect on the maximum stresses, especially in the region of relatively high bending strains (small tube diameters). Note that an approach which considers the effect of the nonlinear stress–strain relationship on the effective Young's modulus, but does not account for this effect on the shift in the centroid of the fiber cross-section, is inconsistent, and results in an overestimation of the maximum tensile stresses and in an underestimation of the maximum compressive stresses.

CONCLUSION

The nonlinear stress–strain relationship in silica materials can have a significant effect on the mechanical behavior of optical fibers experiencing extensive static or dynamic

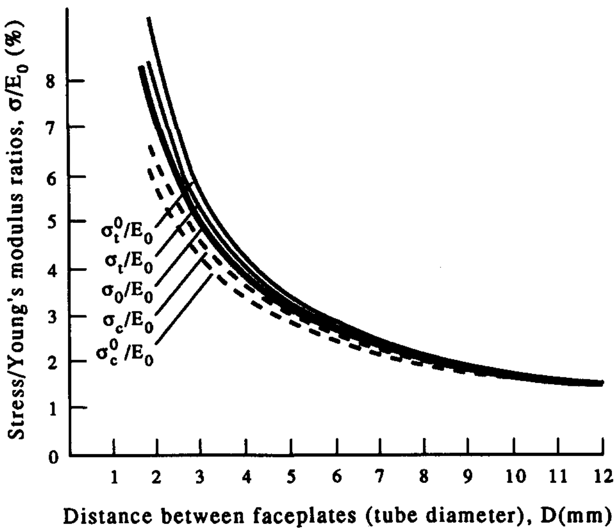

Fig. 4. Maximum stress in a fiber subjected to two-point bending versus distance between faceplates (tube diameter) : σ_t, maximum tensile stress ; σ_c, maximum compressive stress ; σ_t^0, maximum tensile stress without considering the shift in the centroid ; σ_c^0, maximum compressive stress without considering the shift in the centroid, E_0, nominal (low strain) Young's modulus.

loading. The future experimental work should include evaluation of the stress–strain relationship, both in tension and compression, for large strains and for high strength fibers.

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